

1.3.16 Lec 21: Systems: Laplace transforms

The Laplace transform takes real signals $f(t) \in \mathbb{R}$, as a function of real time $t \in \mathbb{R}$, that are *causal*, i.e., strictly zero for negative time ($f(t) = 0$ for $t < 0$), and transforms them into complex functions $F(s) \in \mathbb{C}$ of complex frequency $s = \sigma + \omega j$. As for the Fourier transform, there is the notation $f(t) \leftrightarrow F(s)$.

When a signal is zero for negative time $f(t < 0) = 0$, it is said to be *causal*, and the resulting transform $F(s)$ is then complex analytic over significant regions of the s plane. For a function of time to be causal, time *must* be real ($t \in \mathbb{R}$), since if it were complex, it would lose the order property (thus it could not be causal). It is helpful to emphasize the causal nature of $f(t)u(t)$ by indicating the causal nature, using the Heaviside step function $u(t)$.

Any restriction on a function (e.g., real, causal, periodic, positive real part, etc.) is called a *symmetry property*. There are many forms of symmetry (Section 1.3.17, p. 137). The concept of symmetry is very general and widely used in both mathematics and physics, where it is more generally known as *group theory*. We shall restrict ourselves to only a few very basic symmetries.

Definition of the Laplace transform: The forward and inverse Laplace transforms are

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt \qquad f(t) = \frac{1}{2\pi j} \int_{\sigma_o - \infty j}^{\sigma_o + \infty j} F(s)e^{st} ds \qquad (1.42)$$

$$F(s) \leftrightarrow f(t) \qquad f(t) \leftrightarrow F(s) \qquad (1.43)$$

Here $s = \sigma + j\omega \in \mathbb{C}$ [2 π Hz] is the complex Laplace frequency in radians and $t \in \mathbb{R}$ [s] is the time in seconds.

When dealing with engineering problems it is convenient to separate the *signals* we use from the *systems* that process them. We do this by treating signals, such as a music signal, differently from a system, such as a filter. In general signals may start and end at any time. The concept of causality has no mathematical meaning in signal space. Systems, on the other hand, obey very rigid rules (to assure that they remain physical). These physical restrictions are described in terms of the *network postulates*, which are discussed in Sect. 1.3.17 (p. 137).

There is a question as to why postulates are needed, and which ones are the best choices.⁸³ There may be no answers to these questions, but having a set of postulates is a useful way of thinking about physics.⁸⁴

As discussed in Section 1.4.11 (p. 183), we must use the Cauchy residue theorem (CRT), requiring closure of the contour \mathcal{C} at $\omega j \rightarrow \pm j\infty$

$$\oint_{\mathcal{C}} = \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} + \int_{C_\infty},$$

where the path represented by ' C_∞ ' is a semicircle of infinite radius. For a causal, 'stable' (e.g. doesn't "blow up" in time) signal, all of the poles of $F(s)$ *must* be inside of the Laplace contour, in the left half s -plane.

Hooke's law for a spring states that the force $f(t)$ is proportional to the displacement $x(t)$, i.e., $f(t) = Kx(t)$. The formula for a dash-pot is $f(t) = Rv(t)$, and Newton's famous formula for mass is $f(t) = d[Mv(t)]/dt$, which for constant M is $f(t) = Mdv/dt$.

⁸³https://www.youtube.com/watch?v=JXAfEBbaz_4

⁸⁴<https://www.youtube.com/watch?v=YaUlqXRPMmY>

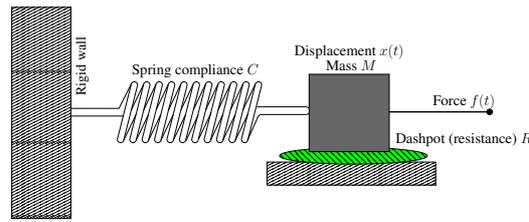


Figure 1.2: Three-element mechanical resonant circuit consisting of a spring, mass and dash-pot (e.g., viscous fluid).

The equation of motion for the mechanical oscillator in Fig. 1.5 is given by Newton's second law; the sum of the forces must balance to zero

$$M \frac{d^2}{dt^2} x(t) + R \frac{d}{dt} x(t) + Kx(t) = f(t). \quad (1.44)$$

These three constants, the mass M , resistance R and stiffness K ($\in \mathbb{R} \geq 0$) are real and non-negative. The dynamical variables are the driving force $f(t) \leftrightarrow F(s)$, the position of the mass $x(t) \leftrightarrow X(s)$ and its velocity $v(t) \leftrightarrow V(s)$, with $v(t) = dx(t)/dt \leftrightarrow V(s) = sX(s)$.

Newton's second law (c1650) is the mechanical equivalent of Kirchoff's (c1850) voltage law (KVL), which states that the sum of the voltages around a loop must be zero. The gradient of the voltage results in a force on a charge (i.e., $F = qE$).

Equation 1.44 may be re-expressed in terms of impedances, the ratio of the force to velocity, once it is transformed into the Laplace frequency domain.

Example: The divergent series

$$e^t u(t) = \sum_0^{\infty} \frac{1}{n!} t^n \leftrightarrow \frac{1}{s-1}$$

is a valid description of $e^t u(t)$, with an unstable pole at $s = 1$. For values of $|x - x_o| < 1$ ($x \in \mathbb{R}$), the analytic function $P(x)$ is said to have a region of convergence (RoC). For cases where the argument is complex ($s \in \mathbb{C}$), this is called the radius of convergence (RoC). We call the region $|s - s_o| > 1$ the region of divergence, and $|s - s_o| = 0$, the singular circle. Typically the underlying function $P(s)$, defined by the series, has a pole on the singular circle for $s \in \mathbb{C}$. One may isolate such poles by moving the expansion point s_o until the RoC approaches zero.

There seems to be a conflict with the time response $f(t) = e^{at} u(t)$, which has a divergent series (unstable pole). I'm not sure how to explain this conflict, other than to point out that $t \in \mathbb{R}$: thus, the series expansion of the diverging exponential is real-analytic, not complex analytic. *First* $f(t)$ has a Laplace transform with a pole at $s = 1$, in agreement with its unstable nature. *Second*, every analytic function must be single valued. This follows from the fact that each term in Eq. 1.50 is single valued. *Third*, analytic functions are "smooth" since they may be differentiated an infinite number of times and the series still converges. There can be no jumps or kinks in such functions.

The key idea that every impedance must be complex analytic and ≥ 0 for $\sigma > 0$ was first proposed by Otto Brune in his PhD at MIT, supervised by Ernst A. Guillemin, an MIT electrical engineering professor who played an important role in the development of circuit theory and was a student of Arnold Sommerfeld.⁸⁵ Other MIT advisers were Norbert Wiener

⁸⁵It must be noted that University of Illinois Prof. 'Mac' Van Valkenburg was arguably more influential in circuit theory during the same period. Mac's books are certainly more accessible, but perhaps less widely cited.

and Vannevar Bush. Brune's primary, but non-MIT, advisor was W. Cauer, also perhaps a student of Sommerfeld, who was well trained in 19th century German mathematics (Brune, 1931b).

Summary: While the definitions of the FT (\mathcal{FT}) and LT (\mathcal{LT}) transforms may appear similar, they are not. The key difference is that the time response of the Laplace transform is causal, leading to a complex analytic frequency response. While the frequency response of the Fourier transform is complex, it is not complex analytic since the frequency ω is real. Fourier transforms do not have poles. These are not superficial differences.

The concept of symmetry is helpful in understanding the many different types of time-frequency transforms. Two fundamental types of symmetry are causality and periodicity.

The Fourier transform \mathcal{FT} characterizes the steady-state response while the Laplace transform \mathcal{LT} characterizes both the transient and steady-state response. Given a causal system response $H(s) \leftrightarrow h(t)$ with input $x(t)$, the output is

$$y(t) = h(t) \star x(t) \leftrightarrow Y(\omega) = H(s) \Big|_{s=j\omega} X(\omega).$$

1.3.17 Lec 22: Ten network postulates

Solutions of differential equations, such as the wave equation, are conveniently described in terms of mathematical properties, which we present here in terms of 10 *network postulates*:

(P1) *Causality* (non-causal/acausal): Causal systems respond when acted upon. Virtually all physical systems obey causality. An example of a causal system is an integrator, which has a response of a step function. Filters are also examples of causal systems. Signals represent acausal responses. They do not have a clear beginning or end, such as the sound of the wind or traffic noise. A causal linear system is typically complex analytic. A nonlinear system can be causal, but not complex analytic.

(P2) *Linearity* (nonlinear): Linear systems obey superposition. Two signals $x(t)$ and $y(t)$ are the inputs to a linear system, producing outputs $x'(t)$ and $y'(t)$. When the inputs are presented together as $ax(t) + by(t)$ with weights $a, b \in \mathbb{C}$, the output will be $ax'(t) + by'(t)$. If either a or b is zero, the corresponding signal is removed from the output.

Nonlinear systems mix the two inputs, thereby producing signals not present in the input. For example, if the inputs to a nonlinear system are two sine waves, the output will contain distortion components, having frequencies not present at the input. An example of a nonlinear system is one that multiplies the two inputs. A second is a diode, which rectifies a signal, letting current flow only in one direction. Most physical systems have some degree of nonlinear response, but this is not always desired. Other systems are designed to be nonlinear, such as the diode example.

(P3) *Passive* (active): An active system has a power source, such as a battery, while a passive system has no power source. While you may consider a transistor amplifier to be active, it is only so when connected to a power source.

(P4) *Real* (complex) time response: Typically systems are "real in, real out." They do not naturally have complex responses (real and imaginary parts). While a Fourier transform takes real inputs and produces complex outputs, this is not an example of a complex time response. P4 is a characterization of the input signal, not its Fourier transform.

1.3.19 Lec 18: Complex analytic mappings (domain-coloring)

One of the most difficult aspects of complex functions of a complex variable is visualizing the mappings from the $z = x + yj$ to $w(z) = u + vj$ planes. For example, $w(z) = \sin(x)$ is trivial when $z = x + yj$ is real (i.e., $y = 0$), because $\sin(x)$ is real. Likewise for the case where $x = 0$,

$$\sin(yj) = \frac{e^{-y} - e^y}{2j} = -j \sinh(y)$$

is pure imaginary. However, the more general case

$$w(z) = \sin(z) \in \mathbb{C}$$

is not easily visualized. And when $u(x, y)$ and $v(x, y)$ are less well known functions, $w(z)$ can be even more difficult to visualize. For example, if $w(z) = J_0(z)$ then $u(x, y), v(x, y)$ are the real and imaginary parts of the Bessel function.

A software solution: Fortunately with computer software today, this problem can be solved by adding color to the chart. An Octave/Matlab script⁸⁶ `zviz.m` has been used to make the charts shown in Fig. 1.3. Such charts are known as *domain-coloring*.⁸⁷

Rather than plotting $u(x, y)$ and $v(x, y)$ separately, domain-coloring allows us to display the entire function on one color chart (i.e., colorized plot). For this visualization we see the complex polar form of $w(s) = |w|e^{j\angle w}$, rather than the 2x2 (four-dimensional) Cartesian graph $w(x + yj) = u(x, y) + v(x, y)j$. On the left is the reference condition, the identity mapping ($w = s$), and on the right the origin has been shifted to the right and up by $\sqrt{2}$.

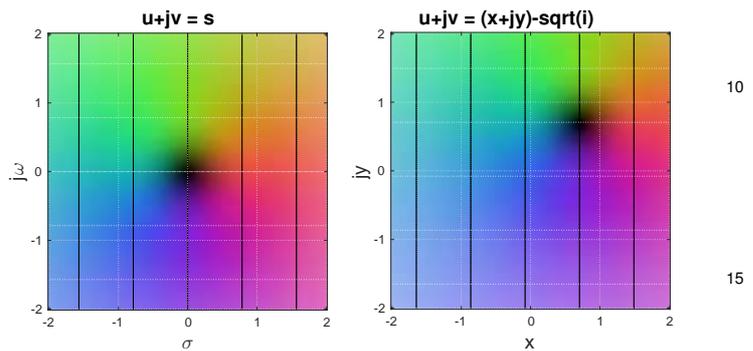


Figure 1.3: Left: Domain-colored map showing the complex mapping from the $s = \sigma + \omega j$ plane to the $w(s) = u(\sigma, \omega) + v(\sigma, \omega)j$ plane. This mapping may be visualized by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase) of the mapping. Right: This shows the $w(z) = z - \sqrt{j}$ plane, shifted to the right and up by $\sqrt{2}/2 = 0.707$. The white and black lines are the iso-real and iso-imaginary contours of the mapping.

Mathematicians typically use abstract (i.e., non-physical) notation $w(z)$, where $w = u + vj$ and $z = x + yj$. Engineers prefer to think in terms of a physical complex impedance $Z(s) = R(s) + jX(s)$, having *resistance* $R(s)$ and *reactance* $X(s)$ [ohms], as a function of the complex Laplace radian frequency $s = \sigma + \omega j$ [rad], as used, for example, with the Laplace transform (Sect. 1.3.16, p. 135).

In Fig. 1.3 we use both notations, with $Z(s) = s$ on the left and $w(z) = z - \sqrt{j}$ on the right, where we show this color code as a 2x2 dimensional domain-coloring graph. Intensity (dark to light) represents the magnitude of the function, while hue (color) represents the phase, where (see Fig. 1.3) red is 0° , sea-green is 90° , blue-green is 135° , blue is 180° , and violet is -90° (or 270°).⁸⁸

⁸⁶<http://jontalle.web.engr.illinois.edu/uploads/298/zviz.zip>

⁸⁷https://en.wikipedia.org/wiki/Domain_coloring

⁸⁸Colors vary depending on both the display medium and the eye.

The function $w = s = |s|e^{j\theta}$ has a dark spot (a zero) at $s = 0$, and becomes brighter away from the origin. On the right is $w(z) = z - \sqrt{j}$, which shifts the zero to $z = \sqrt{j}$. Thus domain-coloring gives the full 2x2 complex analytic function mapping $w(x, y) = u(x, y) + v(x, y)j$, in colorized polar coordinates.

5 **Visualizing complex functions:** The mapping from $z = x + jy$ to $w(z) = u(x, y) + v(x, y)j$ is difficult to visualize because for each point in the domain z , we would like to represent both the magnitude and phase (or real and imaginary parts) of $w(z)$. A good way to visualize these mappings is to use color (hue) to represent the phase and intensity (dark to light) to represent the magnitude. The Octave/Matlab program `zviz.m` has been provided to
 10 do this (see Lecture 17 on the class website).

To use the program in Matlab/Octave, use the syntax `zviz <function of z>` (for example, type `zviz z.^2`). Note the period between z and 2 . This will render a domain-coloring (aka colorized) version of the function. Examples you
 15 can render with `zviz` are given in the comments at the top of the `zviz.m` program. A good example for testing is `zviz z-sqrt(j)`, which will show a dark spot (a zero) at $(1 + 1j)/\sqrt{2} =$
 20 $0.707(1 + 1j)$.

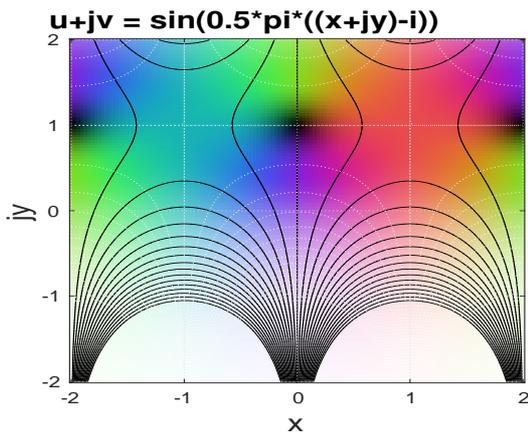


Figure 1.4: Plot of $\sin(0.5\pi(z - i))$.

Example: Figure 1.4 shows a colorized plot of $w(z) = \sin(\pi(z - i)/2)$ resulting from the Matlab/Octave command `zviz sin(pi*(z-i)/2)`. The abscissa (horizontal axis) is the real x axis and the ordinate (vertical axis) is the complex jy axis. The graph is offset along the ordinate axis by $1i$, since the argument $z - i$ causes a shift of the sine function by 1 in the positive imaginary direction.
 25 The visible zeros of $w(z)$ appear as dark regions at $(-2, 1)$, $(0, 1)$, $(2, 1)$. As a function of x , $w(x + 1j)$ oscillates between red (phase is zero degrees), meaning the function is positive and real, and sea-green (phase is 180°), meaning the function is negative and real.

Along the vertical axis, the function is either a $\cosh(y)$ or $\sinh(y)$, depending on x . The
 30 intensity becomes lighter as $|w|$ increases, and white as $w \rightarrow \infty$. The intensity becomes darker as $|w|$ decreases, and black as $w \rightarrow 0$.

Example: Two additional examples are given in Fig. 1.5 using the notation $w(z) = u(x, y) + v(x, y)j$ showing the two complex mappings $w = e^s$ (left) and its inverse $s = \ln(w)$. The exponential is relatively easy to understand because $w(s) = |e^\sigma e^{j\omega}| = e^\sigma$.

35 The red region is where $\omega \approx 0$ in which case $w \approx e^\sigma$. As σ becomes large and negative, $w \rightarrow 0$, thus the entire field becomes dark on the left. The field is becoming light on the right where $w = e^\sigma \rightarrow \infty$. If we let $\sigma = 0$ and look along the ω axis, we see that the function is changing phase, sea-green (90°) at the top and violet (-90°) at the bottom.

In the right panel note the zero for $\ln(w) = \ln|w| + j\omega$ at $w = 1$. The root of the log function
 40 is $\log(w_r) = 0$, $w_r = 1$, $\phi = 0$, since $\log(1) = 0$. More generally, the log of $w = |w|e^{j\phi}$ is $s = \ln|w| + j\phi$. Thus $s(w)$ can be zero only when the angle of w is zero.

The $\ln(w)$ function has a *branch cut* along the $\phi = 180^\circ$ axis. As one crosses over the cut, the phase goes above 180°, and the plane changes to the next *sheet* of the log function. The only sheet with a zero is the principle value, as shown. For all others, the log function is

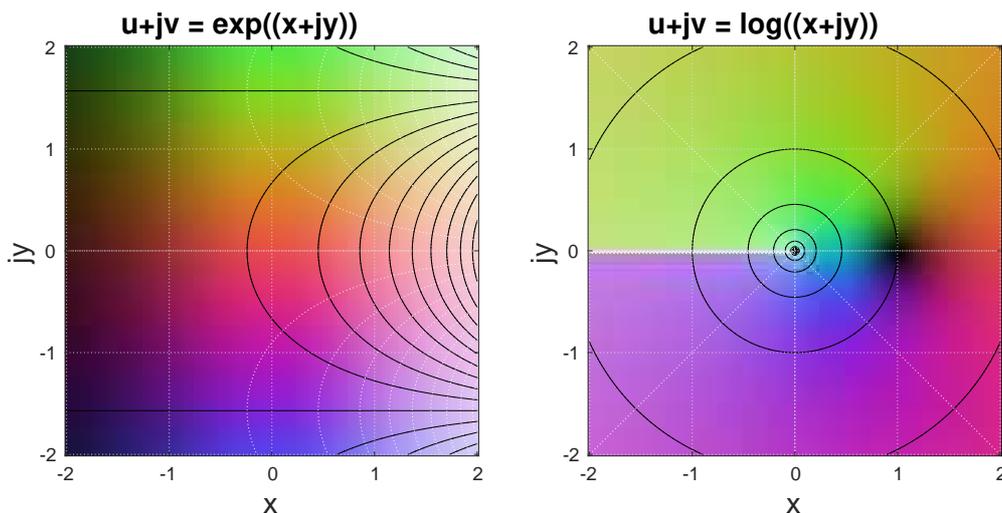


Figure 1.5: This domain-color map allows one to visualize complex mappings by the use of intensity (light/dark) to indicate magnitude, and color (hue) to indicate angle (phase). The white and black lines are the iso-real and iso-imaginary contours of the mapping. Left: This figure shows the domain-color map for the complex mapping from the $z = x + yj$ plane to the $w(z) = u + vj = e^{x+yj}$ plane, which goes to zero as $x \rightarrow -\infty$, causing the domain-color map to become dark for $x < -2$. The white and black lines are always perpendicular because e^z is complex analytic everywhere. Right: This shows the principal value of the inverse function $u(x, y) + v((x, y)j) = \log(x + yj)$, which has a zero (dark) at $z = 1$, since there $\log(1) = 0$ (the imaginary part is zero). Note the branch cut from $x = [0, -\infty)j$. On branches other than the one shown, there are no zeros since the phase ($\angle z = 2\pi n \in \mathbb{Z}$) is not zero. n is called the branch index.

either increasing or decreasing monotonically, and there is no zero, as seen for sheet 0 (the one showing in Fig. 1.5).

1.4 Stream 3a: Scalar (i.e., Ordinary) Differential Equations

Stream 3 is ∞ , a concept which typically means unbounded (immeasurably large), but in the case of calculus, ∞ means *infinitesimal* (immeasurably small), since taking a limit requires small numbers. Taking a limit means you may never reach the target, a concept that the Greeks called *Zeno's paradox* (Stillwell, 2010, p. 76).

When speaking of the class of *ordinary* (versus *vector*) differential equations, the term *scalar* is preferable, since the term “ordinary” is vague, if not a meaningless label. There is a special subset of fundamental theorems for scalar calculus, all of which are about integration, as summarized in Table 1.8 (p. 153), starting with Leibniz's theorem. These will be discussed below, and more extensively in Lec. 1.4.2 (p. 151) and Sections 1.4 (p. 146) and 1.5 (p. 146).

Chronological history post 16th century

16th Bombelli 1526–1572

17th Galileo 1564–1642, Kepler 1571–1630, Newton 1642–1727 Principia 1687; Mersenne; Huygens; Pascal; Fermat, Descartes (analytic geometry); and the three Bernoullis (Jakob, Johann & son Daniel)

18th Euler 1707–1783 (Student of Johann Bernoulli), d'Alembert 1717–1783, Lagrange 1736–1833, Laplace 1749–1827, Fourier 1768–1830, Gauss 1777–1855 Cauchy 1789–1857,

19th Helmholtz 1821–1894, Kelvin 1824–1907, Helmholtz 1821–1894, Kirchhoff 1824–1887, Riemann 1826–1866, Maxwell 1831–1879, Rayleigh 1842–1919 Heaviside 1850–1925,

20th Hilbert 1862–1942, Einstein 1879–1955

TimeLine

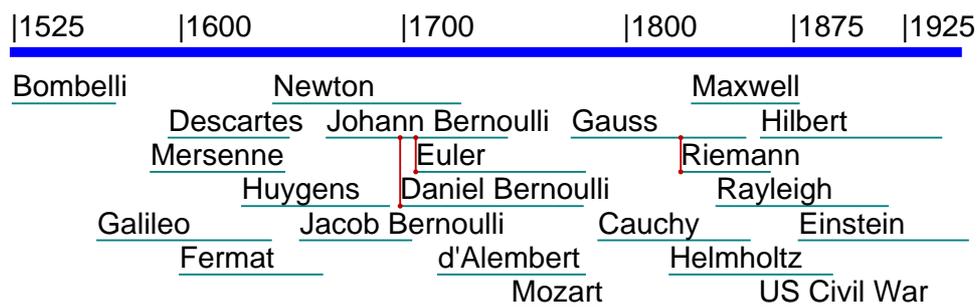


Figure 1.6: Final overview of the timeline for the four centuries from the 16th and 20th CE covering Bombelli to Einstein. Mozart and the US Civil War are indicated along the bottom, for orientation.

Following the integral theorems on scalar calculus are those on vector calculus, without which there could be no understanding of Maxwell's equations. Of these, the *fundamental theorem of complex calculus* (aka, Helmholtz decomposition), Gauss's law and Stokes's theorem form the cornerstone of modern vector field analysis. These theorems allow one to connect the differential (point) and macroscopic (integral) relationships. For example, Maxwell's equations may be written as either vector differential equations, as shown by Heaviside (along with

Gibbs and Hertz),⁸⁹ or in integral form. It is helpful to place these two forms side-by-side, to fully appreciate their significance. To understand the differential (microscopic) view, one must understand the integral (macroscopic) view. These are presented in Section 1.5.13 (p. 230) and Fig. 1.24 (p. 233).

The beginning of modern mathematics

As outlined in Fig. 1.6, mathematics as we know it today began in the 16th to 18th centuries, arguably starting with Galileo, Descartes, Fermat, Newton, the Bernoulli family, and Euler. Galileo was formidable, due to his fame, fortune, and his “successful” stance against the powerful Catholic establishment. His creativity in scientific circles was certainly well known due to his many skills and accomplishments. Descartes and Fermat were at the forefront of merging algebra and geometry. While Fermat kept meticulous notebooks, he did not publish, and tended to be secretive. Thus Descartes’s contributions were more widely acknowledged, but not necessarily deeper.

Regarding the development of calculus, much was yet to be done by Newton and Leibniz, using term-by-term integration of functions based on Taylor series representation. This was a powerful technique, but as stated earlier, incomplete because **the Taylor series can only represent single-valued functions, within the RoC**. But more importantly, **Newton (and others) failed to recognize the powerful generalization to complex analytic functions**. The first major breakthrough was Newton’s publication of *Principia* (1687), and the second was Riemann (1851), advised by Gauss.

Following Newton’s lead, the secretive and introverted behavior of the typical mathematician dramatically changed with the Bernoulli family (Fig. 1.6). The oldest brother Jacob taught his much younger brother Johann, who then taught his son Daniel. But Johann’s star pupil was Euler. Euler first mastered all the tools and then published, with a prolificacy previously unknown.

Euler and the circular functions: The first major task was to understand the family of analytic circular functions, e^x , $\sin(x)$, $\cos(x)$, and $\log(x)$, a task begun by the Bernoulli family, but mastered by Euler. Euler sought relations between these many functions, some of which may not be thought of as being related, such as the log and sin functions. The connection that may “easily” be made is through their complex Taylor series representation (Eq. 1.51, p. 88). By the manipulation of the analytic series representations, the relationship between e^x , and the $\sin(x)$ and $\cos(x)$, was precisely captured with the relation

$$e^{j\omega} = \cos(\omega) + j\sin(\omega), \quad (1.45)$$

and its analytic inverse (Greenberg, 1988, p. 1135)

$$\tan^{-1}(z) = \frac{1}{2j} \ln \left(\frac{1j - z}{1j + z} \right). \quad (1.46)$$

Exercise: Starting from Eq. 1.45, derive Eq. 1.46. **Solution:** Let $z(\omega) = \tan \omega$, then

$$z(\omega) = \frac{\sin \omega}{\cos \omega} = \tan(\omega) = -j \frac{e^{j\omega} - e^{-j\omega}}{e^{j\omega} + e^{-j\omega}} = -j \frac{e^{2j\omega} - 1}{e^{2j\omega} + 1}. \quad (1.47)$$

⁸⁹https://en.wikipedia.org/wiki/History_of_Maxwell%27s_equations

Solving this for $e^{2\omega j}$ gives

$$e^{2\omega j} = \frac{1 + zj}{1 - zj}. \quad (1.48)$$

Taking the square root results in the analytic relationship between $e^{\omega j}$ and $j \tan \omega$

$$e^{\omega j} = \pm \sqrt{\frac{1 + zj}{1 - zj}}.$$

Taking the log, and using the definition of $\omega(z) = \tan^{-1}(z)$, we obtain Eq. 1.46.

These equations are the basis of transmission lines (TL) and the Smith chart. Here $z(\omega)$ is the TL's input impedance and Eq. 1.48 is the *reflectance*.

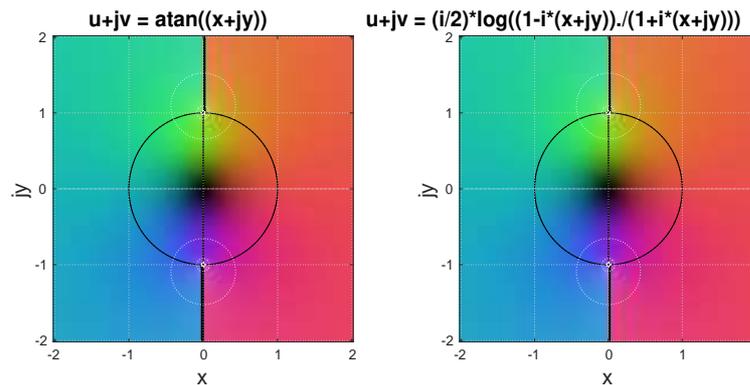


Figure 1.7: Colorized plots of $\omega(z) = \tan^{-1}(z)$ and $\omega(z) = \frac{i}{2} \ln(1 - iz)/(1 + iz)$, verifying they are the same complex analytic function.

While many high school students memorize Euler's relation, it seems unlikely they appreciate the significance of *complex analytic functions* (Eq. 1.62, p. 93).

A brief history of complex analytic functions: Newton famously ignored imaginary numbers, and called them imaginary in a disparaging (pejorative) way. Given Newton's prominence, his view certainly must have attenuated interest in complex algebra, even though it had been previously quantified by Bombelli in 1525, likely based on his serendipitous finding of Diophantus's book *Arithmetic* in the Vatican library.

Euler derived his relationships using real power series (i.e., real analytic functions). While Euler was fluent with $j = \sqrt{-1}$, he did not consider functions to be complex analytic. That concept was first explored by Cauchy almost a century later. The missing link to the concept of complex analytic is the definition of the derivative with respect to the complex argument

$$F'(s) = \frac{dF(s)}{ds}, \quad (1.49)$$

where $s = \sigma + \omega j$, without which the complex analytic Taylor coefficients may not be defined.

Euler did not appreciate the role of complex analytic functions because these were first discovered well after his death (1783) by Augustin-Louis Cauchy (1789–1857).

1.4.1 Lec 17: Riemann Sphere: 3^d extension of chord and tangent method

15

Once algebra was formulated by c830 CE, mathematicians were able to expand beyond the limits set by geometry on the real plane, and the verbose descriptions of each problem in prose

(Stillwell, 2010, p. 93). The geometry of Euclid’s *Elements* had paved the way, but after 2000 years, the addition of the language of algebra changed everything. The analytic function was a key development, heavily used by both Newton and Euler. Also the investigations of Cauchy made important headway with his work on complex variables. Of special note was integration and differentiation in the complex plane of complex analytic functions, which is the topic of stream 3.

It was Riemann, working with Gauss in the final years of Gauss’s life, who made the breakthrough, with the concept of the *extended complex plane*.⁹⁰ This concept was based on the composition of a line with the sphere, similar to the derivation of Euclid’s formula for Pythagorean triplets (Fig. I.1, p. 338). While the importance of the extended complex plane was unforeseen, it changed analytic mathematics forever, along with the physics it supported. It unified and thus simplified many important integrals, to the extreme. The basic idea is captured by the *fundamental theorem of complex integral calculus* (Table 1.8, p. 153).

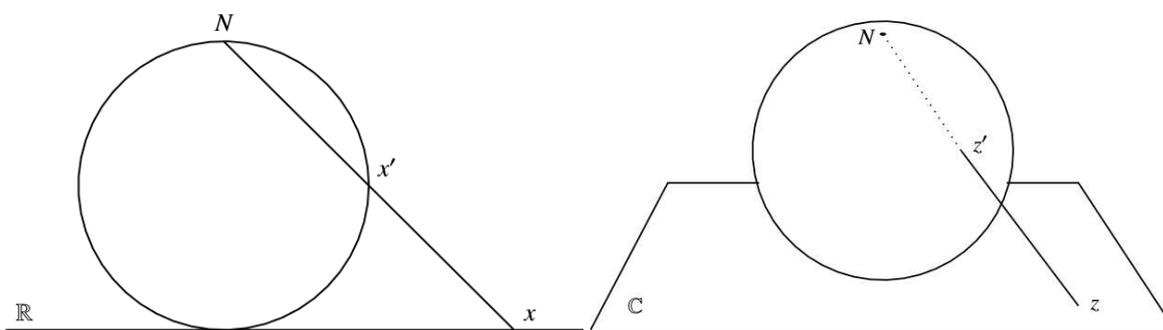


Figure 1.8: The left panel shows how the real line may be composed with the circle. Each real x value maps to a corresponding point x' on the unit circle. The point $x \rightarrow \infty$ maps to the north pole N . This simple idea may be extended with the composition of the complex plane with the unit sphere, thus mapping the plane onto the sphere. As with the circle, the point on the complex plane $z \rightarrow \infty$ maps onto the north pole N . This construction is important because while the plane is open (does not include $z \rightarrow \infty$), the sphere is analytic at the north pole. Thus the sphere defines the closed extended plane. Figure adapted from Stillwell (2010, pp. 299-300).

The idea is outlined in Fig. 1.8. On the left is a circle and a line. The difference between this case and the derivation of the Pythagorean triplets is that the line starts at the north pole, and ends on the real $x \in \mathbb{R}$ axis at point x . At point x' , the line cuts through the circle. Thus the mapping from x to x' takes every point on \mathbb{R} to a point on the circle. For example, the point $x = 0$ maps to the south pole (not indicated). To express x' in terms of x one must compose the line and the circle, similar to the composition used in Fig. I.1 (p. 338). The points on the circle, indicated here by x' , require a traditional polar coordinate system, having a unit radius and an angle defined between the radius and a vertical line passing through the north pole. When $x \rightarrow \infty$ the point $x' \rightarrow N$, known as the *point at infinity*. But this idea goes much further, as shown on the right half of Fig. 1.8.

Here the real tangent line is replaced by a tangent complex plane $z \in \mathbb{C}$, and the complex puncture point $z' \in \mathbb{C}$, in this case on the complex sphere, called the *extended complex plane*. This is a natural extension of the chord/tangent method on the left, but with significant consequences. The main difference between the complex plane z and the extended complex plane, other than the coordinate system, is what happens at the north pole. The point at $|z| = \infty$ is

⁹⁰“Gauss did lecture to Riemann but he was only giving elementary courses and there is no evidence that at this time he recognized Riemann’s genius.” Then “In 1849 he [Riemann] returned to Göttingen and his Ph.D. thesis, supervised by Gauss, was submitted in 1851.” <http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Riemann.html>

not defined on the plane, whereas on the sphere, the point at the north pole is simply another point, like every other point on the sphere.

Open vs. closed sets: Mathematically the plane is said to be an *open set*, since the limit $z \rightarrow \infty$ is not defined, whereas on the sphere, the point z' is a member of a *closed set*, since the north pole is defined. The distinction between an open and closed set is important, because the closed set allows the function to be complex analytic at the north pole, which it cannot be on the plane (since the point at infinity is not defined).

The z plane may be replaced with another plane, say the $w = F(z) \in \mathbb{C}$ plane, where w is some function F of $z \in \mathbb{C}$. For the moment we shall limit ourselves to *complex analytic functions* of z , namely $w = F(z) = u(x, y) + v(x, y)j = \sum_{n=0}^{\infty} c_n z^n$.

In summary, given a point $z = x + yj$ on the open complex plane, we map it to $w = F(z) \in \mathbb{C}$, the complex $w = u + vj$ plane, and from there to the closed extended complex plane $w'(z)$. The point of doing this is that it allows us to allow the function $w'(z)$ to be analytic at the north pole, meaning it can have a convergent Taylor series at the point at infinity $z \rightarrow \infty$. Since we have not yet defined $dw(z)/dz$, the concept of a complex Taylor series remains undefined.

Bilinear transformation

In mathematics the *bilinear transformation* has special importance because it is linear in its action on both the input and output variables. Since we are engineers we shall stick with the engineering terminology. But if you wish to read about this on the internet, be sure to also search for the mathematical term, Möbius transformation.

When a point on the complex plane $z = x + yj$ is composed with the bilinear transform $(a, b, c, d \in \mathbb{C})$, the result is $w(z) = u(x, y) + v(x, y)j$

$$w = \frac{az + b}{cz + d}. \quad (1.50)$$

The transformation from $z \rightarrow w$ is a cascade of four independent compositions:

1. translation ($w = z + b$: $a = 1, b \in \mathbb{C}, c = 0, d = 1$),
2. scaling ($w = |a|z$: $a \in \mathbb{R}, b = 0, c = 0, d = 1$),
3. rotation ($w = \frac{a}{|a|}z$: $a \in \mathbb{C}, b = 0, c = 0, d = |a|$) and
4. inversion ($w = \frac{1}{z}$: $a = 0, b = 1, c = 1, d = 0$).

Each of these transformations is a special case of Eq. 1.50, with the inversion the most complicated. I highly recommend a video showing the effect of the bilinear (Möbius) transformation on the plane.⁹¹

The bilinear transformation is the most general way to move the expansion point in a complex analytic expansion. For example, starting from the harmonic series, the bilinear transform gives

$$\frac{1}{1-w} = \frac{1}{1 - \frac{az+b}{cz+d}} = \frac{cz+d}{(c-a)z + (d-b)}.$$

⁹¹<https://www.youtube.com/watch?v=0z1fIsUNhO4>

The RoC is transformed from $|w| < 1$ to $|(az - b)/(cz - d)| < 1$. An interesting application might be in moving the expansion point until it is on top of the nearest pole, so that the RoC goes to zero. This might be a useful way of finding a pole, for example.

When the extended plane (Riemann sphere) is analytic at $z = \infty$, one may take the derivatives there, defining the Taylor series with the expansion point at ∞ . When the bilinear transformation rotates the Riemann sphere, the point at infinity is translated to a finite point on the complex plane, revealing the analytic nature at infinity. A second way to transform the point at infinity is by the bilinear transformation $\zeta = 1/z$, mapping a zero (or pole) at $z = \infty$ to a pole (or zero) at $\zeta = 0$. Thus this construction of the Riemann sphere and the Möbius (bilinear) transformation allows us to understand the point at infinity, and treat it like any other point. If you felt that you never understood the meaning of the point at ∞ (likely), this should help.

1.4.2 Lec 23: Fundamental theorems of calculus

History of the fundamental theorem of calculus: In some sense, the story of calculus begins with the *fundamental theorem of calculus* (FTC), also known generically as *Leibniz's formula*. The simplest integral is the length of a line $L = \int_0^L dx$. If we label a point on a line as $x = 0$ and wish to measure the distance to any other point x , we form the line integral between the two points. If the line is straight, this integral is simply the Euclidean length given by the difference between the two ends (Eq. 1.3.7, p. 108).

If $F(\chi) \in \mathbb{R}$ describes a height above the line $\chi \in \mathbb{R}$, then $f(x)$

$$f(x) - f(0) = \int_{\chi=0}^x F(\chi) d\chi \quad (1.51)$$

may be viewed as the *anti-derivative* of $F(\chi)$. Here χ is a dummy variable of integration. Thus the area under $F(\chi)$ only depends on the difference of the area evaluated at the end points. It makes intuitive sense to view $f(x)$ as the anti-derivative of $F(\chi)$.

This property of the area as an integral over an interval, only depending on the end points, has important consequences in physics in terms of conservation of energy, allowing for important generalizations. For example, as long as $\chi \in \mathbb{R}$, one may let $F(\chi) \in \mathbb{C}$ with no loss of generality, due to the linearity property (P1, p. 137) of the integral.

If $f(x)$ is analytic (Eq. 1.50, p. 87), then

$$F(x) = \frac{d}{dx} f(x) \quad (1.52)$$

is an exact real differential. It follows that $F(x)$ is analytic. This is known as the *fundamental theorem of (real) calculus* (FTC). Thus Eq. 1.52 may be viewed as an *exact real differential*. This is easily shown by evaluating

$$\frac{d}{dx} f(x) = \lim_{\delta \rightarrow 0} \frac{f(x + \delta) - f(x)}{\delta} = F(x),$$

starting from the *anti-derivative* Eq. 1.51. If $f(x)$ is not analytic then the limit may not exist, so this is a necessary condition.

There are many important variations on this very basic theorem (i.e., Sect. 1.4, p. 146). For example, the limits could depend on time. Also when taking Fourier transforms, the integrand depends on both time $t \in \mathbb{R}$ and frequency $\omega \in \mathbb{R}$ via a complex exponential “kernel” function $e^{\pm j\omega t} \in \mathbb{C}$, which is analytic in both t and ω .

The fundamental theorems of complex calculus:

The *fundamental theorem of complex calculus* (FTCC) states (Greenberg, 1988, p. 1197) that for any *complex analytic function* $F(s) \in \mathbb{C}$ with $s = \sigma + \omega j \in \mathbb{C}$,

$$f(s) - f(s_o) = \int_{s_o}^s F(\zeta) d\zeta. \quad (1.53)$$

Equations 1.51 and 1.53 differ because the path of the integral is complex. Thus the line integral is over $s \in \mathbb{C}$ rather than a real integral over $\chi \in \mathbb{R}$. The *fundamental theorem of complex calculus* (FTCC) states that the integral only depends on the end points, since

$$F(s) = \frac{d}{ds} f(s). \quad (1.54)$$

Comparing exact differentials Eq. 1.49 (FTCC) and Eq. 1.52 (FTC), we see that $f(s) \in \mathbb{C}$ must be *complex analytic*, and have a Taylor series in powers in $s \in \mathbb{C}$. It follows that $F(s)$ is also
5 complex analytic.

Complex analytic functions: The definition of a *complex analytic function* $F(s)$ of $s \in \mathbb{C}$ is that the function may be expanded in a Taylor series (Eq. 1.61, p. 93) about an *expansion point* $s_o \in \mathbb{C}$. This definition follows the same logic as the FTC. Thus we need a definition for the coefficients $c_n \in \mathbb{C}$, which most naturally follow from Taylor's formula

$$c_n = \frac{1}{n!} \left. \frac{d^n}{ds^n} F(s) \right|_{s=s_o}. \quad (1.55)$$

The requirement that $F(s)$ have a Taylor series naturally follows by taking derivatives with respect to s at s_o . *The problem is that both integration and differentiation of functions of complex Laplace frequency $s = \sigma + \omega j$ have not yet been defined.*

Thus the question is: What does it mean to take the derivative of a function $F(s) \in \mathbb{C}$,
10 $s = \sigma + \omega j \in \mathbb{C}$, with respect to s , where s defines a plane rather than a real line? We learned how to form the derivative on the real line. Can the same derivative concept be extended to the complex plane?

The answer is affirmative. The question may be resolved by applying the rules of the real derivative when defining the derivative in the complex plane. However, for the complex case,
15 there is an issue regarding direction. Given any analytic function $F(s)$, is the partial derivative with respect to σ different from the partial derivative with respect to ωj ? For complex analytic functions, the FTCC states that the integral is independent of the path in the s plane. *Based on the chain rule, the derivative must also be independent of direction at s_o .* This directly follows from the FTCC. *If the integral of a function of a complex variable is to be independent of the*
20 *path, the derivative of a function with respect to a complex variable must be independent of the direction.* This follows from Taylor's formula, Eq. 1.55, for the coefficients of the complex analytic formula.

The Cauchy-Riemann conditions: The FTC defines the area as an integral over a real differential ($dx \in \mathbb{R}$), while the FTCC relates an integral over a complex function $F(s) \in \mathbb{C}$, along
25 a complex interval (i.e., path) ($ds \in \mathbb{C}$). For the FTC the area under the curve only depends on the end points of the anti-derivative $f(x)$. But what is the meaning of an "area" along a complex path? The *Cauchy-Riemann conditions* provide the answer.

Table 1.8: Summary of the fundamental theorems of integral calculus, each of which deals with integration. There are at least two main theorems related to scalar calculus, and three more for vector calculus. [tab:FundThms](#)

Name	Mapping	p.	Description
<i>Leibniz (FTC)</i>	$\mathbb{R}^1 \rightarrow \mathbb{R}^0$	151	Area under a real curve
<i>Cauchy (FTCC)</i>	$\mathbb{C}^1 \rightarrow \mathbb{R}^0$	151	Area under a complex curve
<i>Cauchy's theorem</i>	$\mathbb{C}^1 \rightarrow \mathbb{C}^0$	170	Close integral over analytic region is zero
<i>Cauchy's integral formula</i>	$\mathbb{C}^1 \rightarrow \mathbb{C}^0$	170	Fundamental theorem of complex integral calculus
<i>residue theorem</i>	$\mathbb{C}^1 \rightarrow \mathbb{C}^0$	170	Residue integration

1.4.3 Lec 24: Cauchy-Riemann conditions

For the integral of $Z(s) = R(\sigma, \omega) + X(\sigma, \omega)j$ to be independent of the path, the derivative of $Z(s)$ must be independent of the direction of the derivative. As we show next, this leads to a pair of equations known as the *Cauchy-Riemann* conditions. This is an important generalization of Eq. 1.1 (p. 20), which goes from real integration ($x \in \mathbb{R}$) to complex integration ($s \in \mathbb{C}$) based on lengths, thus on area.

To define

$$\frac{d}{ds}Z(s) = \frac{d}{ds} [R(\sigma, \omega) + jX(\sigma, \omega)],$$

take partial derivatives of $Z(s)$ with respect to σ and $j\omega$, and equate them:

$$\frac{\partial Z}{\partial \sigma} = \frac{\partial R}{\partial \sigma} + j \frac{\partial X}{\partial \sigma} \quad \equiv \quad \frac{\partial Z}{\partial j\omega} = \frac{\partial R}{\partial j\omega} + j \frac{\partial X}{\partial j\omega}.$$

This says that a horizontal derivative, with respect to σ , is equivalent to a vertical derivative, with respect to $j\omega$. Taking the real and imaginary parts gives the two equations

$$\text{CR-1: } \frac{\partial R(\sigma, \omega)}{\partial \sigma} = j \frac{\partial X(\sigma, \omega)}{\partial \omega j} \quad \text{and} \quad \text{CR-2: } \frac{\partial R(\sigma, \omega)}{\partial \omega j} = -j \frac{\partial X(\sigma, \omega)}{\partial \sigma}, \quad (1.56)$$

known as the *Cauchy-Riemann (CR) conditions*. The j cancels in CR-1, but introduces a $j^2 = -1$ in CR-2. They may also be written in polar coordinates ($s = re^{j\theta}$) as

$$\frac{\partial R}{\partial r} = \frac{1}{r} \frac{\partial X}{\partial \theta} \quad \text{and} \quad \frac{\partial X}{\partial r} = -\frac{1}{r} \frac{\partial R}{\partial \theta}.$$

If you are wondering what would happen if we took a derivative at 45 degrees, then we only need to multiply the function by $e^{j\pi/4}$. But doing so will not change the derivative. Thus we may take the derivative in any direction by multiplying by $e^{j\theta}$, and the CR conditions will not change.

The CR conditions are necessary conditions that the integral of $Z(s)$, and thus its derivative, be independent of the path, expressed in terms of conditions on the real and imaginary parts of Z . This is a very strong condition on $Z(s)$, which follows assuming that $Z(s)$ may be written as a Taylor series in s :

$$Z(s) = Z_0 + Z_1s + \frac{1}{2}Z_2s^2 + \dots, \quad (1.57)$$

where $Z_n \in \mathbb{C}$ are complex constants given by the Taylor series formula (Eq. 1.55, p. 152). As with the real Taylor series, there is the convergence condition, that $|s| < 1$, called the *radius of*

convergence (RoC). This is an important generalization of the region of convergence (R0C) for real $s = x$.

Every function that may be expressed as a *Taylor series* in $s - s_o$ about point $s_o \in \mathbb{C}$ is said to be *complex analytic* at s_o . This series, *which must be single valued*, is said to converge within a *radius of convergence* (RoC). This highly restrictive condition has significant physical consequences. For example, every impedance function $Z(s)$ obeys the CR conditions over large regions of the s plane, including the entire right half-plane (RHP) ($\sigma > 0$). This condition is summarized by the Brune condition $\Re\{Z(\sigma > 0)\} \geq 0$ (Section 1.4.4, Eq. 1.65, p. 157). When the CR condition is generalized to volume integrals, it is called *Green's theorem*, used heavily in the solution of boundary value problems in engineering and physics (Kusse and Westwig, 2010). Sections 1.4 (p. 146) and 1.5 (p. 191) depend heavily on these concepts.

We may merge these equations into a pair of second-order equations by taking a second round of partials. Specifically, eliminating the real part $R(\sigma, \omega)$ of Eq. 1.56 gives

$$\frac{\partial^2 R(\sigma, \omega)}{\partial \sigma \partial \omega} = \frac{\partial^2 X(\sigma, \omega)}{\partial^2 \omega} = -\frac{\partial^2 X(\sigma, \omega)}{\partial^2 \sigma}, \quad (1.58)$$

which may be compactly written as $\nabla^2 X(\sigma, \omega) = 0$. Eliminating the imaginary part gives

$$\frac{\partial^2 X(\sigma, \omega)}{\partial \omega \partial \sigma} = \frac{\partial^2 R(\sigma, \omega)}{\partial^2 \sigma} = -\frac{\partial^2 R(\sigma, \omega)}{\partial^2 \omega}, \quad (1.59)$$

which may be written as $\nabla^2 R(\sigma, \omega) = 0$.

In summary, for a function $Z(s)$ to be complex analytic, the derivative dZ/ds must be independent of direction (path), which requires that the real and imaginary parts of the function obey Laplace's equation, i.e.,

$$\text{CR-3: } \nabla^2 R(\sigma, \omega) = 0 \quad \text{and} \quad \text{CR-4: } \nabla^2 X(\sigma, \omega) = 0. \quad (1.60)$$

The CR equations are easier to work with because they are first-order, but the physical intuition is best understood by noting two facts (1) the derivative of a complex analytic function is independent of its direction, and (2) the real and imaginary parts of the function both obey Laplace's equation. Such relationships are known as *harmonic functions*.⁹²

As we shall see in the next few lectures, complex analytic functions must be smooth since every analytic function may be differentiated an infinite number of times, within the RoC. The magnitude must attain its maximum *and* minimum on the boundary. For example, when you stretch a rubber sheet over a jagged frame, the height of the rubber sheet obeys Laplace's equation. Nowhere can the height of the sheet rise above or below its value at the boundary.

Harmonic functions define *conservative fields*, which means that energy (like a volume or area) is conserved. The work done in moving a mass from a to b in such a field is conserved. If you return the mass from b back to a , the energy is retrieved, and zero net work has been done.

1.4.4 Lec 25: Complex Analytic functions and Brune Impedance

It is rarely stated that the variable that we are integrating over, either x (space) or t (time), is real ($x, t \in R$), since that fact is implicit, due to the physical nature of the formulation of the

⁹²When the function is the ratio of two polynomials, as in the cases of the Brune impedance, they are also related to Möbius transformations, also known as *bi-harmonic* operators.

integral. But this intuition must be refined once complex numbers are included with $s \in \mathbb{C}$, where $s = \sigma + \omega j$.

That time and space are real variables is more than an assumption: it is a requirement that follows from the real *order property*. Real numbers have order. For example, if $t = 0$ is now (the present), then $t < 0$ is the past and $t > 0$ is the future. The order property of time and space allows one to order these along a real axis. To have time travel, time and space would need to be complex (they are not), since if the space axis were complex, as in frequency s , the order property would be invalid. It follows that if we require order, time and space must be real ($t, x \in \mathbb{R}$). Interestingly, it was shown by d'Alembert (1747) that time and space are related by the pure delay due to the wave speed c_o . To obtain a solution to the governing wave equation, which d'Alembert first proposed for sound waves, $x, t \in \mathbb{R}^3$ may be combined as functions of

$$\zeta_{\pm} = t \pm x/c_o,$$

where c [m/s] is the phase velocity of the waves. The d'Alembert solution to the wave equation, describing waves on a string under tension, is

$$u(x, t) = f(t - x/c_o) + g(t + x/c_o), \quad (1.61)$$

which describes the transverse velocity (or displacement) of two independent waves $f(\zeta_-)$, $g(\zeta_+)$ on the string, which represent forward and backward traveling waves.⁹³ For example, starting with a string at rest, if one displaces the left end, at $x = 0$, by a step function $u(t)$, then that step displacement will propagate to the right as $u(t - x/c_o)$, arriving at location x_o [m], at time x_o/c_o [s]. Before this time, the string will not move to the right of the wave-front, at x_o [m], and after t_o [s] it will have displacement 1. Since the wave equation obeys superposition (postulate P1, p. 137), it follows that the “plane-wave” eigenfunctions of the wave equation for $\mathbf{x}, \mathbf{k} \in \mathbb{R}^3$ are given by

$$\psi_{\pm}(\mathbf{x}, t) = \delta(t \mp \mathbf{k} \cdot \mathbf{x}) \leftrightarrow e^{st \pm j\mathbf{k} \cdot \mathbf{x}}, \quad (1.62)$$

where $|\mathbf{k}| = 2\pi/|\lambda| = \omega/c_o$ is the *wave number*, $|\lambda|$ is the wavelength, and $s = \sigma + \omega j$, the Laplace frequency.

When propagation losses are considered, we must replace the *wave number* $j\mathbf{k}$ with a complex analytic vector wave number $\boldsymbol{\kappa}(s) = \mathbf{k}_r(s) + j\mathbf{k}(s)$, which is denoted as either the *complex propagation function*, *propagation function*, or the *dispersion relation*. The vector complex propagation function is a key generalization of the scalar wave number $k = 2\pi/\lambda$.

Forms of loss include viscosity and radiation. Important examples include acoustic plane waves, electromagnetic wave propagation, antenna theory, and the more complex case of 3D electron wave propagating in crystals (i.e., silicon), where electrons and EM waves are in a state of equilibrium.

Electrons and photons are simply different EM states, where $\boldsymbol{\kappa}(\mathbf{x}, s)$ describes the crystal's dispersion relations as functions of both frequency and direction, famously known as *Brillouin zones*. Dispersion is a property of the medium such that the wave velocity is a function of frequency and direction, as in silicon.⁹⁴ Informed discussions on the history of this topic may be found in Brillouin (1953).

⁹³d'Alembert's solution is valid for functions that are not differentiable, such as $\delta(t - c_o x)$.

⁹⁴In case you missed it, I'm suggesting is that photons (propagating waves) and electrons (evanescent waves) are different wave “states.” The difference is the medium, which determines the dispersion relation (Papasimakis *et al.*, 2018).

Complex impedance functions

Conservation of energy (or power) is a cornerstone of modern physics. It may first have been under consideration by Galileo Galilei (1564-1642) and Marin Mersenne (1588-1648). Today the question is not whether it is true, but why. Specifically, what is the physics behind con-
 5 servation of energy? Surprisingly, the answer is straightforward, based on its definition and the properties of impedance. Recall that the power is the product of the force and flow, and impedance is their ratio.

The power is given by the product of two variables, sometimes called *conjugate* variables, the force and the flow. In electrical terms, these are voltage (force) ($v(t) \leftrightarrow V(\omega)$) and current (flow) ($i(t) \leftrightarrow I(\omega)$); thus, the electrical power at any instant of time is

$$\mathcal{P}(t) = v(t)i(t). \quad (1.63)$$

The total energy $E(t)$ is the integral of the power, since $\mathcal{P}(t) = dE/dt$. Thus if we start with all the elements at rest (no currents or voltages), then the energy as a function of time is always positive

$$E(t) = \int_0^t \mathcal{P}(\tau) d\tau \geq 0, \quad (1.64)$$

and is simply the total energy applied to the network (Van Valkenburg, 1964a, p. 376). Since the voltage and current are related by either an impedance or an admittance, conservation of energy depends on the property of impedance. From Ohm's law and P1 (every impedance is causal)

$$v(t) = z(t) \star i(t) = \int_{\tau=0}^t z(\tau) i(t - \tau) d\tau \leftrightarrow V(s) = Z(s)I(s).$$

Needs work!

Example: Let $i(t) = \delta(t)$. Then $|w|^2(\tau) = i(t) \star i(t) = \delta(\tau)$. Thus

$$I_{xx}(t) = \int_{\tau=0}^t z(\tau) |w|^2(\tau) d\tau = \int_{\tau=0}^t z(\tau) \delta(\tau) d\tau = \int_0^t z(\tau) d\tau.$$

The Brune impedance always has the form $z(t) = r_o \delta(t) + \zeta(t)$. The surge impedance is defined as

$$r_o = \int_{0^-}^{\infty} z(t) dt.$$

The integral of the reactive part ($\zeta(t)$) is always zero, since the reactive part cannot not store energy.

Perhaps easier to visualize is when working in the frequency domain where the total energy, equal to the integral of the real part of the power, is

$$\frac{1}{s} \Re VI = \frac{1}{2s} (V^* I + V I^*) = \frac{1}{2s} (Z^* I^* I + Z I I^*) = \frac{1}{s} \Re Z(s) |I|^2 \geq 0.$$

10 Formally this is related to a positive definite operator where the positive resistance forces the definiteness, which is sandwiched between the current.

In conclusion, conservation of energy is totally dependent on the properties of the impedance. Thus one of the most important and obvious applications of complex functions of a complex variable is the impedance function. This seems to be the ultimate example of the
 15 FTC, applied to $z(t)$, in the name of conservation of energy.

Poles and zeros of PR functions must be first degree: We conjecture that this proof also requires that the poles and the zeros of the impedance function be simple (only first degree). Second degree poles would have a reactive “secular” response of the form $h(t) = t \sin(\omega_k t + \phi)u(t)$, and these terms would not average to zero, depending on the phase, as is required of an impedance. As a result, only single degree poles would be possible.⁹⁵ Furthermore, when the impedance is the ratio of two polynomials, where the lower degree polynomial is the derivative of the higher degree one, then the poles and zeros must alternate. This is a well-known property of the Brune impedance that has never been adequately explained except for very special cases (Van Valkenburg, 1964b, p. 104-107). I suspect that no one has ever reported an impedance having second degree poles and zeros as that would be rare impedance. Network analysis books never report 2nd degree poles and zeros in their impedance functions. Nor has there ever been any guidance as to where the poles and zeros might lie in the *left-hand s plane*. Understanding the exact relationships between pairs of poles and zeros, to assure that the real part of the impedance is real, would resolve a longstanding unsolved problem (Van Valkenburg, 1964b).

Every impedance must obey conservation of energy (P3): The impedance function $Z(s)$ has resistance R and reactance X as a function of complex frequency $s = \sigma + j\omega$. From the causality postulate (P1) of Sections 1.3.17 and K.5.1 (p. 443), $z(t < 0) = 0$. Every impedance is defined by a Laplace transform pair

$$z(t) \leftrightarrow Z(s) = R(\sigma, \omega) + jX(\sigma, \omega),$$

with $R, X \in \mathbb{R}$.

According to Postulate P3 (Sect. 1.3.17, p. 137), a system is passive if it does not contain a power source. Drawing power from an impedance violates conservation of energy. This propriety is also called *positive-real*, which was defined by Brune (1931b,a)

$$\Re\{Z(s \geq 0)\} \geq 0. \quad (1.65)$$

Positive-real systems cannot draw more power than is stored in the impedance.⁹⁶ The region $\sigma \leq 0$ is called the *left half s plane* (LHP), and the complementary region $\sigma > 0$ is called the *right half s plane* (RHP). According to the Brune condition the real part of every impedance must be non-negative in the RHP.

It is easy to construct examples of second-order poles or zeros in the RHP, such that P3 is violated. Thus P3 implies that the impedance may *not* have more than simple (first-order) poles and zeros, strictly in the LHP. But there is yet more: These poles and zero in the LHP must have order, to meet the minimum phase condition. This minimum phase condition is easily stated

$$\angle Z(s) < \angle s,$$

but difficult to prove.

There seems to be no proof that second-order poles and zeros (e.g., second-order roots) are not allowed. However such roots would violate a requirement that the poles and zeros must alternate on the $\sigma = 0$ axis, which follows from P3. In the complex plane the concept of ‘alternate’ is not defined (complex numbers cannot be ordered). What *has* been proved is that if the poles are on the real or imaginary axis, they must alternate, leading to simple poles and

⁹⁵Secular terms result from second degree poles since $u(t) * u(t) = tu(t) \leftrightarrow 1/s$.

⁹⁶Does this condition hold for the LHP $\sigma < 0$? It does for Eq. 1.67.

zeros. The restriction on poles is sufficient, but not necessary, as $Z(s) = 1/\sqrt{s}$ is a *physical realizable* (PR) impedance, but is less than a first-degree pole (Kim and Allen, 2013). The corresponding condition in the LHP, and its proof, remains elusive (Van Valkenburg, 1964a).

As an example, a series resistor R_o and capacitor C_o have an impedance given by (Table G.1, p. 286)

$$Z(s) = R_o + 1/sC_o \leftrightarrow R_o\delta(t) + \frac{1}{C_o}u(t) = z(t), \quad (1.66)$$

with constants $R_o, C_o \in \mathbb{R} > 0$. In mechanics an impedance composed of a dash-pot (damper) and a spring has the same form. A resonant system has an inductor, resistor and a capacitor, with an impedance given by (Table G.2, p. 287)

$$Z(s) = \frac{sC_o}{1 + sC_oR_o + s^2C_oM_o} \leftrightarrow C_o \frac{d}{dt} (c_+e^{s_+t} + c_-e^{s_-t}) = z(t), \quad (1.67)$$

which is a second degree polynomial with two complex resonant frequencies s_{\pm} . When $R_o > 0$ these roots are in the left half s plane, with $z(t) \leftrightarrow Z(s)$.

Systems (networks) containing many elements, and transmission lines, can be much more complicated, yet still have a simple frequency domain representation. This is the key to understanding how these physical systems work, as will be described below.

Complex analytic functions: To solve a differential equation, or integrate a function, Newton used the Taylor series to integrate one term at a time. However, he only used real functions of a real variable, due to the fundamental lack of appreciation of the complex analytic function. This same method is how one finds solutions to scalar differential equations today, but using an approach that makes the solution method less obvious. Rather than working directly with the Taylor series, today we use the complex exponential, since the complex exponential is an eigenfunction of the derivative

$$\frac{d}{dt}e^{st} = se^{st}.$$

Since e^{st} may be expressed as a Taylor series, having coefficients $c_n = 1/n!$, in some real sense the modern approach is a compact way of doing what Newton did. Thus every linear constant coefficient differential equation in time may be simply transformed into a polynomial in complex Laplace frequency s , by looking for solutions of the form $A(s)e^{st}$, transforming the differential equation into a polynomial $A(s)$ in complex frequency. For example,

$$\frac{d}{dt}f(t) + af(t) \leftrightarrow (s + a)F(s).$$

The root of $A(s_r) = s_r + a = 0$ is the eigenvalue of the differential equation. A powerful tool for understanding the solutions of differential equations, both scalar and vector, is to work in the Laplace frequency domain. The Taylor series has been replaced by e^{st} , transforming Newton's real Taylor series into the complex exponential eigenfunction. In some sense, these are the same method, since

$$e^{st} = \sum_{n=0}^{\infty} \frac{(st)^n}{n!}. \quad (1.68)$$

Taking the derivative with respect to time gives

$$\frac{d}{dt}e^{st} = se^{st} = s \sum_{n=0}^{\infty} \frac{(st)^n}{n!}, \quad (1.69)$$

which is also complex analytic. Thus if the series for $F(s)$ is valid (i.e., it converges), then its derivative is also valid. This was a very powerful concept, exploited by Newton for real functions of a real variable, and later by Cauchy and Riemann for complex functions of a complex variable. The key question here is: Where does the series fail to converge? If it does not converge, the representation fails. This is the main message behind the FTCC (Eq. 1.53). 5

The FTCC (Eq. 1.49) is formally the same as the FTC (Eq. 1.52) (Leibniz formula), the key (and significant) difference being that the argument of the integrand $s \in \mathbb{C}$. Thus this integration is a line integral in the complex plane. One would naturally assume that the value of the integral depends on the path of integration.

But, according to FTCC, it does not. In fact they are clearly distinguishable from the FTC. 10 And the reasoning is the same. If $F(s) = df(s)/ds$ is complex analytic (i.e., has a power series $f(s) = \sum_k c_k s^k$, with $f(s), c_k, s \in \mathbb{C}$), then it may be integrated, and *the integral does not depend on the path*. At first blush, this is sort of amazing. The key is that $F(s)$ and $f(s)$ must be complex analytic, which means they are differentiable. This all follows from the Taylor series formula Eq. 1.55 (p. 152) for the coefficients of the complex analytic series. For Eq. 1.53 15 to hold, the derivatives must be independent of the direction, as discussed in Section 1.4.3. The concept of a complex analytic function therefore has eminent consequences, in the form of several key theorems on complex integration discovered by Cauchy (c1820).

The use of the complex Taylor series generalizes the functions it describes, with unpredictable consequences, as nicely shown by the domain coloring diagrams presented in Section 20 1.3.19 (p. 143). Cauchy's tools of complex integration were first exploited in physics by Sommerfeld (1952) to explain the onset transients in waves, as explained in detail in Brillouin (1960, Chap. 3).

Up to 1910, when Sommerfeld first published his results using complex analytic signals and saddle point integration in the complex plane, there was a poor understanding of the im- 25 plications of the causal wave-front. It would be reasonable to say that his insights changed our understanding of wave propagation, for both light and sound. Sadly this insight has not been fully appreciated, even to this day. If you question my summary, please read Brillouin (1960, Chap. 1).

The full power of the complex analytic function was first appreciated by Bernard Riemann 30 (1826-1866) in his University of Göttingen PhD thesis of 1851, under the tutelage of Carl Friedrich Gauss (1777-1855), and drawing heavily on the work of Cauchy.

The key definition of a complex analytic function is that it has a Taylor series representa- 35 tion over a region of the complex frequency plane $s = \sigma + j\omega$, that converges in a *region of convergence* (RoC) about the expansion point, with a radius determined by the nearest pole of the function. A further surprising feature of all analytic functions is that within the RoC, the inverse of that function also has a complex analytic expansion. Thus given $w(s)$, one may also determine $s(w)$ to any desired accuracy, critically depending on the RoC.

1.4.5 Lec 26: Multi-valued functions

In the field of mathematics there seems to have been a tug-of-war regarding the basic defini- 40 tion of the concept of a function. The accepted definition today seems to be a single-valued mapping from the domain to the codomain (or range). This makes the discussion of multi-valued functions somewhat tedious. In 1851 Riemann (working with Gauss) seems to have resolved this problem for the natural set of multi-valued functions by introducing the concept of single-valued *sheets*, delineated by *branch-cuts*. 45

Two important examples of multi-valued functions are the circle $z^2 = x^2 + y^2$ and $w =$